Differentiation: Quotient Rule

The **Quotient Rule** is used when we want to differentiate a function that may be regarded as a quotient of two simpler functions.

If our function f can be expressed as $f(x) = \dfrac{g(x)}{h(x)}$, where g and h are simpler functions, then the Quotient Rule may be stated as

$$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{h(x)^2} \qquad \text{or} \qquad \frac{df}{dx}(x) = \frac{\frac{dg}{dx}(x)h(x) - g(x)\frac{dh}{dx}(x)}{h(x)^2}.$$

In particular note that there is a minus sign in the numerator. Either writing this as a plus or getting the $g'(x)h(x)$ and the $g(x)h'(x)$ the wrong way around can be a source of errors.

Example 1: Find the derivative of $f(x) = \dfrac{x^2+x+1}{x^3-x^2+x-1}$.

Solution 1: In this case we let our functions g and h be

$$g(x) = x^2 + x + 1 \qquad \text{and} \qquad h(x) = x^3 - x^2 + x - 1.$$

Then

$$g'(x) = 2x + 1 \qquad \text{and} \qquad h'(x) = 3x^2 - 2x + 1.$$

Next, using the Quotient Rule, we see that the derivative of f is

$$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{h(x)^2} = \frac{(2x+1)(x^3-x^2+x-1) - (x^2+x+1)(3x^2-2x+1)}{(x^3-x^2+x-1)^2}.$$

It is possible to multiply the expression in the numerator out and simplify it a little, but whether or not you need to do this will depend on the course you are taking and what your lecturer wants you to do.

Example 2: Find the derivative of $f(x) = \tan x$.

Solution 2: At first sight we don't appear to have a quotient here, however we know that the tangent is defined as $\tan x = \dfrac{\sin x}{\cos x}$, so we do really have a quotient.

So let our functions g and h be

$$g(x) = \sin x \qquad \text{and} \qquad h(x) = \cos x.$$

Then

$$g'(x) = \cos x \qquad \text{and} \qquad h'(x) = -\sin x.$$

Next, using the Quotient Rule, we see that the derivative of f is

$$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{h(x)^2} = \frac{\cos x \cos x - \sin x(-\sin x)}{(\cos x)^2} = \frac{\cos^2 x + \sin^2 x}{(\cos x)^2} = \frac{1}{(\cos x)^2} = \sec^2 x.$$

Example 3: Find the derivative of $f(x) = \dfrac{(x^2-1)(2\cos 3x)}{\ln x}$.

Solution 3: While we do obviously have a quotient here, we also have a product in the numerator, so before we can make any progress in differentiating f, we have to use the Product Rule to differentiate the function in the numerator. If you are unsure how to use the Product Rule to differentiate the function $g(x) = (x^2-1)(2\cos 3x)$, then please see the worksheet **Differentiation: Product Rule**, where it is differentiated in Example 2.
Now, let our functions g and h be

$$g(x) = (x^2-1)(2\cos 3x) \qquad \text{and} \qquad h(x) = \ln x.$$

Then, using the result from **Differentiation: Product Rule**, Example 2,

$$g'(x) = 4x\cos 3x - 6(x^2-1)\sin 3x \qquad \text{and} \qquad h'(x) = \frac{1}{x}.$$

Next, using the Quotient Rule, we see that the derivative of f is

$$\begin{aligned} f'(x) &= \frac{g'(x)h(x) - g(x)h'(x)}{h(x)^2} \\ &= \frac{(4x\cos 3x - 6(x^2-1)\sin 3x)\ln x - (x^2-1)(2\cos 3x)\frac{1}{x}}{\left(\frac{1}{x}\right)^2} \\ &= x^2(4x\cos 3x - 6(x^2-1)\sin 3x)\ln x - x(x^2-1)(2\cos 3x). \end{aligned}$$

Example 4: Find the derivative of $f(x) = \dfrac{5}{x^3+x^2+x+1}$.

Solution 4: Let our functions g and h be

$$g(x) = 5 \qquad \text{and} \qquad h(x) = x^3+x^2+x+1.$$

Then

$$g'(x) = 0 \qquad \text{and} \qquad h'(x) = 3x^2+2x+1.$$

Next, using the Quotient Rule, we see that the derivative of u is

$$f'(x) = \frac{0(x^3+x^2+x+1) - 5(3x^2+2x+1)}{(x^3+x^2+x+1)^2} = -\frac{5(3x^2+2x+1)}{(x^3+x^2+x+1)^2}.$$

While this does give the correct answer, it is slightly easier to differentiate this function using the **Chain Rule**, and this is covered in another worksheet.

Example 5: Find the derivative of $f(x) = \dfrac{x^2-1}{x+1}$.

Solution 5: In this case we let our functions g and h be

$$g(x) = x^2-1 \qquad \text{and} \qquad h(x) = x+1.$$

Then

$$g'(x) = 2x \qquad \text{and} \qquad h'(x) = 1.$$

Next, using the Quotient Rule, we see that the derivative of f is

$$f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{h(x)^2} = \frac{(2x)(x+1) - (x^2-1)(1)}{(x-1)^2} = \frac{x^2+2x+1}{(x+1)^2} = \frac{(x+1)^2}{(x+1)^2} = 1.$$

Note that we also have $f(x) = \dfrac{x^2-1}{x+1} = \dfrac{(x-1)(x+1)}{x+1} = x-1$, so $f'(x) = 1$.
So it is always a good idea to see if you can simplify the original function in some way at the start, since this might lead to an easier solution.